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Journal of Geometry and Physics 47 (2003) 484–496
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Quantum diffusion on manifolds

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Received 5 July 2002; received in revised form 11 November 2002

Abstract

We investigate non-degenerate diffusion processes on an arbitrary manifold, the dynamics of which arise from a principle of least action for a Lagrangian consisting of a kinetic term quadratic in the forward drift of the process and a local potential. The equation governing the action emerges as a stochastic Hamilton–Jacobi condition and is expressed in terms of the geometry determined by the Levi-Civita connection of the diffusion tensor. It is argued that there are essentially two dynamical structures for the rate of change of the drift in the presence of a local potential, consistent with the requirement of time reversal symmetry. In both cases a conserved energy is identified. An alternative wave function and associated operator description reveals a complex structure in the dynamical equations, thus extending the earlier results of Nelson on the stochastic treatment of the Schrödinger equation.

MSC: 58J68; 81P20

JGP SC: Quantum mechanics

Keywords: Quantum diffusion

1. Introduction

In this paper we discuss a class of non-degenerate diffusion processes on an arbitrary manifold $\mathcal{M}$ that possess time-symmetric conservative dynamics derivable from a local potential function on $\mathcal{M}$. We present the functional forms of the dynamics explicitly and show how these arise from a stochastic principle of least action for different choices of kinetic Lagrangian. Our treatment is closely aligned to the development of stochastic mechanics initiated by Nelson [1,2] and extends this by discussing the general case of time-symmetric conservative dynamics, the nature of the conserved energy of the process, the associated wave function and operator formalisms and emergent complex structures. Our analysis is carried

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doi:10.1016/S0393-0440(03)00004-4

out using the natural metrical geometry supplied by the inverse of the (non-degenerate) contravariant diffusion tensor. Essentially, two distinct dynamical structures emerge, both of which are derivable from a stochastic principle of least action. The relevant Lagrangian consists of a kinetic part minus a local potential, and the two dynamical structures are tied to the specification of the kinetic Lagrangian in terms of the forward drift of the process. The notion of acceleration is not unique for a stochastic process, owing to the distinct notions of forward and backward drift velocities and time derivatives that are conditioned with respect to the past and future. In this respect the choice of dynamics amounts to a relationship between the local potential and a specific choice of acceleration quantity that arises as a combination of past or future conditioned time derivatives of the forward or backward drifts. The dynamics can also be expressed as the (non-linear) stochastic Hamilton–Jacobi equation for the action. In both cases the exponential of the action gives rise to a wave function description and associated operator formalism in terms of which the dynamics is linear. For one choice of kinetic Lagrangian the stochastic principle of least action is known [1,2] to lead to diffusion dynamics equivalent to the Schrödinger wave equation of quantum theory [3]. The remaining choice, although similar in appearance in terms of wave function dynamics, is nevertheless distinct for a given potential, and has not been discussed previously in the context of stochastic differential and quantum geometry. Although the dynamical law is time-symmetric, in contrast to the Schrödinger case the wave function description has an inherent time asymmetry with respect to Hermitian conjugation, which lends itself to stochastic control theory problems in which a final value for the action is specified. The wave function description leads us to the remarkable result that, for either choice of dynamics, a complex structure is present acting on the Hilbert space of states $\mathcal{H}$. It is shown how the choice of dynamics can be captured by the specification of this complex structure [4,5] via its action on the (time derivative of the) wave function, and an associated Hamiltonian operator. The existence of complex structures in the wave function formalism (already present in the Schrödinger equation) for the general choice of conservative diffusion dynamics constitutes a significant new result.

This paper is organised as follows. We begin in Section 2 with an exposition of the elements of stochastic differential geometry for non-degenerate diffusion processes on a manifold $\mathcal{M}$ that are necessary for our discussion. Section 3 discusses the kinematics of diffusion and introduces the action functional as an integral over a sample path of the diffusion process, and defines a stochastic principle of least action giving rise to a relationship between the action and Lagrangian density. The main body of new results appear in Sections 4 and 5. In Section 4 we posit the dynamical equations in vector form and show how these can be derived from the stochastic principle of least action. Section 5 discusses the emergence of complex structures in a wave function description of both dynamical structures, and in this context addresses the nature of conserved energy, stationary states and the principle of superposition.

The reader should be familiar with the basic elements of differential geometry [6], diffusion theory [7] and the Ito stochastic calculus [8]. Tensorial indices shown in roman bold type denote the coordinate basis representation, while those in plain italic are abstract indices that serve to indicate to which space the relevant geometrical object belongs. We adopt the Einstein summation convention throughout [9]. Indices from $\{i, j, k, \dots\}$ refer to $\mathcal{M}$ while those from $\{a, b, c, \dots\}$, $\{\alpha, \beta, \gamma, \dots\}$ refer to the Hilbert space of states $\mathcal{H}$.

2. Stochastic differential geometry

We introduce the diffusion process X_t^i on a manifold $\mathcal{M}$ and the associated contravariant diffusion tensor σ^{ij}, which in the non-degenerate case has inverse σ_{ij} that provides a natural metrical structure on $\mathcal{M}$. The Levi-Civita connection of this metric ∇_i is introduced for the purpose of differential geometric operations, and has the simplifying property that it annihilates the diffusion tensor. In this geometry it is natural to work with the Ito drift of the process, whose transformation properties are tensorial with respect to $(\mathcal{M}, \sigma_{ij})$. The situation in this regard is contrasted with the case of the Kolmogoroff forward drift obtained by regarding the process as existing on $\mathbb{R}^n$ [10]. The current treatment has the advantage over previous work [1,2] that each geometrical operation may be expressed in the coordinate free abstract index notation, as elucidated, for example in [11].

Consider a continuous time diffusion process X_t^i on a manifold $\mathcal{M}$ of dimension n taken to satisfy the (time independent) Ito stochastic differential equation [8]

$$\mathrm{d}X_t^{\mathbf{i}} = \beta^{\mathbf{i}}(X_t^i)\,\mathrm{d}t + \sum_I \sigma_I^i(X_t^i)\,\mathrm{d}W_t^I, \tag{2.1}$$

where $\{W_t^I\}$ are a collection of n independent Wiener processes [8] satisfying $\mathrm{d}W^J = \delta^{IJ}\mathrm{d}t$. The Wiener summation term in the contravariant diffusion tensor $\sigma^{\mathbf{ij}}$ determined by $\mathrm{d}X_t^{\mathbf{i}}\,\mathrm{d}X_t^{\mathbf{j}} = \sigma^{\mathbf{ij}}\,\mathrm{d}t$[1] then satisfies $\sigma^{\mathbf{ij}} = \sum_I \sigma_I^{\mathbf{i}}\sigma_I^{\mathbf{j}}$. We distinguish between two notions of drift on the manifold as follows [10]. Regarding $X_t^{\mathbf{i}}$ as a process on a coordinate patch of $\mathbb{R}^n$ we have the Kolmogoroff mean forward drift [10] defined as

$$\beta^{\mathbf{i}}(x,t) = \lim_{\delta t\to 0} \mathbf{E}_{t,x}\left[\frac{X_{t+\delta t}^{\mathbf{i}} - X_t^{\mathbf{i}}}{\delta t}\right], \tag{2.2}$$

and the mean backward drift

$$\tilde{\beta}^{\mathbf{i}}(x,t) = \lim_{\delta t\to 0} \mathbf{E}_{t,x}\left[\frac{X_t^{\mathbf{i}} - X_{t-\delta t}^{\mathbf{i}}}{\delta t}\right] \tag{2.3}$$

for $\delta t > 0$ in a coordinate chart $\{x^{\mathbf{i}}\}$ that is normal with respect to the Euclidean metrical geometry δ_{ij} of $\mathbb{R}^n$ [9] and where $\mathbf{E}_{t,x}$ is the expectation conditional on the value x of the process at time t. Denoting the Levi-Civita connection of δ_{ij} by ∂_i, the probability density $\rho_{\mathcal{E}}$ with respect to the Euclidean volume measure on $\mathbb{R}^n$ satisfies the Fokker–Planck equation [7]

$$\frac{\partial\rho_{\mathcal{E}}}{\partial t} + \partial_i(\rho_{\mathcal{E}}\beta^i) = \frac{1}{2}\partial_i\partial_j(\sigma^{ij}\rho_{\mathcal{E}}), \tag{2.4}$$

and the corresponding backward equation

$$\frac{\partial\rho_{\mathcal{E}}}{\partial t} + \partial_i(\rho_{\mathcal{E}}\tilde{\beta}^i) = -\frac{1}{2}\partial_i\partial_j(\sigma^{ij}\rho_{\mathcal{E}}). \tag{2.5}$$

For a non-degenerate diffusion we write $\sigma^{ij}\sigma_{jk} = \delta_k^i$, with $\sigma := \det\{\sigma_{ij}\} \neq 0$. In this way the inverse σ_{ij} supplies the natural metrical geometry of $\mathcal{M}$ from a diffusion point of view, in

[1] The absence of expectation on the left-hand side here is the feature that rules out discontinuous jump processes.

that no extra metrical ingredient is required in the construction of a diffusion on $\mathcal{M}$. In the subsequent sections we shall work mainly with the Levi-Civita connection ∇_i of σ^{ij}, whose components are given by $\nabla_i V^j = \partial_i V^j + \Gamma^i_{jk} V^k$ with $\Gamma^i_{jk} = (1/2)\sigma^{ip}(\partial_j \sigma_{pk} + \partial_k \sigma_{pj} - \partial_p \sigma_{jk})$ [9]. Thus ∇_i annihilates the diffusion tensor, which shall be a useful property in the calculations that follow. In the spirit of the σ_{ij} metrical geometry it is more convenient to define the *Ito* forward and backward drifts in an analogous way to (2.2) and (2.3), except that $\{x^{\mathbf{i}}\}$ now refers to a coordinate chart that is normal with respect to the Levi-Civita connection ∇_i of the σ_{ij} metrical geometry. This drift quantity has the virtue that its components change in a tensorial way under coordinate transformations. The probability density ρ (for the same process) with respect to the invariant σ volume measure $\sqrt{\sigma}\,\mathrm{d}^n x^i$ is related to the previous density by $\rho_{\mathcal{E}} = \rho\sqrt{\sigma}$ and satisfies the covariant Fokker–Planck equations[2]

$$\frac{\partial \rho}{\partial t} + \nabla_i(\rho b^i) = \frac{1}{2}\sigma^{ij}\nabla_i \nabla_j \rho, \tag{2.6a}$$

$$\frac{\partial \rho}{\partial t} + \nabla_i(\rho \tilde{b}^i) = -\frac{1}{2}\sigma^{ij}\nabla_i \nabla_j \rho. \tag{2.6b}$$

Observe that, since the connection ∇_i is designed to annihilate σ, the positioning of the σ^{ij} term on the right-hand side above has no effect, unlike in (2.4) and (2.5). This feature will prove useful in the calculations that follow. Comparison of the two forms of the Fokker–Planck equation yields[3]

$$b^{\mathbf{i}} = \beta^{\mathbf{i}} - \frac{1}{2\sqrt{\sigma}}\partial_{\mathbf{j}}(\sqrt{\sigma}\sigma^{\mathbf{ij}}). \tag{2.7}$$

This relationship can be re-expressed in terms of the Christoffel symbol Γ as follows. From the identity $\Gamma^{\mathbf{k}}_{\mathbf{kj}} = \partial_{\mathbf{j}}(\log\sqrt{\sigma})$ [9] we obtain for the second term on the right-hand side $(1/\sqrt{\sigma})\partial_{\mathbf{j}}(\sqrt{\sigma}\sigma^{\mathbf{ij}}) = \partial_{\mathbf{j}}\sigma^{\mathbf{ij}} + \sigma^{\mathbf{ij}}\Gamma^{\mathbf{k}}_{\mathbf{kj}}$. We then define $\Gamma^{\mathbf{i}} := \sigma^{\mathbf{jk}}\Gamma^{\mathbf{i}}_{\mathbf{jk}} = -\partial_{\mathbf{j}}\sigma^{\mathbf{ij}} - \sigma^{\mathbf{ij}}\Gamma^{\mathbf{k}}_{\mathbf{kj}}$ and thus (2.7) can be written as $b^{\mathbf{i}} = \beta^{\mathbf{i}} + (1/2)\Gamma^{\mathbf{i}}$, in accordance with [2]. We deduce the transformation law for $\beta^{\mathbf{i}}$ from the Ito equation $\mathrm{d}X^{\mathbf{i}} = \beta^{\mathbf{i}}\,\mathrm{d}t + \mathrm{d}w^{\mathbf{i}}_t + \mathrm{o}(\mathrm{d}t)$ which implies that $\mathrm{d}X^{\mathbf{i}'} = (\partial x^{\mathbf{i}'}/\partial x^{\mathbf{i}})(\beta^{\mathbf{i}}\,\mathrm{d}t + \mathrm{d}w^{\mathbf{i}}_t) + (1/2)(\partial^2 x^{\mathbf{i}'}/\partial x^{\mathbf{i}}\partial x^{\mathbf{j}})\,\mathrm{d}w^{\mathbf{i}}_t\,\mathrm{d}w^{\mathbf{j}}_t + \mathrm{o}(\mathrm{d}t)$. Taking $\mathbf{E}_t$ of this equation and using the definition (2.2) in general coordinates yields $\beta^{\mathbf{i}'} = (\partial x^{\mathbf{i}'}/\partial x^{\mathbf{i}})\beta^{\mathbf{i}} + (1/2)(\sigma^{\mathbf{ij}})(\partial^2 x^{\mathbf{i}'}/\partial x^{\mathbf{i}}\partial x^{\mathbf{j}})$. Recalling the transformation for the components of the Christoffel symbol $\Gamma^{\mathbf{i}'}_{\mathbf{j}'\mathbf{k}'} = P^{\mathbf{i}'}_{\mathbf{i}} P^{\mathbf{j}}_{\mathbf{j}'} P^{\mathbf{k}}_{\mathbf{k}'}\Gamma^{\mathbf{i}}_{\mathbf{jk}} - P^{\mathbf{j}}_{\mathbf{j}'} P^{\mathbf{k}}_{\mathbf{k}'}\partial_{\mathbf{j}} P^{\mathbf{i}'}_{\mathbf{k}}$, where $P^{\mathbf{i}'}_{\mathbf{i}} = \partial x^{\mathbf{i}'}/\partial x^{\mathbf{i}}$ [6,9] it follows that $\Gamma^{\mathbf{i}'} = (\partial x^{\mathbf{i}'}/\partial x^{\mathbf{i}})\Gamma^{\mathbf{i}} - \sigma^{\mathbf{jk}}(\partial^2 x^{\mathbf{i}'}/\partial x^{\mathbf{j}}\partial x^{\mathbf{k}})$. Hence the non-tensorial terms in the transformations for the components $\beta^{\mathbf{i}}$, $\Gamma^{\mathbf{i}}$ cancel in (2.7) so that the components $b^{\mathbf{i}}$ transform vectorially. Observe that for a non-degenerate diffusion on $\mathcal{M}$ a sufficient condition for the Kolmogoroff and Ito drifts to coincide at a given point P is that $\partial_i \sigma_{jk}$ (or equivalently Γ^i_{jk}) vanish there. (The converse does not hold, however, since it is possible for Γ^i to vanish with $\partial_i \sigma_{jk}$ not equivalent to zero.)

[2] Adding these two equations yields the equation of continuity familiar from fluid dynamics [12] $\partial\rho/\partial t + \nabla_i(\rho v^i) = 0$.

[3] The comparison in fact shows that the ρ-weighted left- and right-hand sides of the equation below have equal divergences. Since this holds for arbitrary ρ independent of b, β, σ the identity follows.

The situation in respect of the abstract index notation can be summarised as follows (cf. Eqs. (7.1) and (7.2) in [13]). The components of the abstract covariant derivative $\nabla_j \xi^i$ of a vector in the tangent space to $\mathcal{M}$, $\xi^i \in T\mathcal{M}$, can be expressed as

$$\delta^j_{\mathbf{j}} \delta^{\mathbf{i}}_i (\nabla_j \xi^i) = \frac{\partial \xi^{\mathbf{i}}}{\partial x^{\mathbf{j}}} \Gamma^{\mathbf{i}}_{\mathbf{jk}} \xi^{\mathbf{k}}, \tag{2.8}$$

where $\delta^i_{\mathbf{i}}$ is a coordinate basis on a patch of $\mathcal{M}$ and $\delta^{\mathbf{i}}_i$ its dual. Thus from (2.1) and (2.7) we have

$$\mathrm{d}x^{\mathbf{i}} + \frac{1}{2} \Gamma^{\mathbf{i}}_{\mathbf{jk}} \sigma^{\mathbf{jk}} \,\mathrm{d}t - \sum_i \sigma^{\mathbf{i}}_i \,\mathrm{d}W^i = b^{\mathbf{i}} \,\mathrm{d}t. \tag{2.9}$$

This enables us to introduce the abstract Ito differential

$$\mathrm{d}x^i = \delta^i_{\mathbf{i}} (\mathrm{d}x^{\mathbf{i}} + \tfrac{1}{2} \sigma^{\mathbf{jk}} \Gamma^{\mathbf{i}}_{\mathbf{jk}} \,\mathrm{d}t), \tag{2.10}$$

and hence (2.1) becomes the abstract geometrical equation $\mathrm{d}x^i = b^i \,\mathrm{d}t + \mathrm{d}w^i_t$. The vector $\beta^i = \delta^i_{\mathbf{i}} \beta^{\mathbf{i}}$ is then a genuine tensorial object on $\mathcal{M}$ but one whose definition via (2.2) is tied to the choice of a normal coordinate chart with respect to the Levi-Civita connection ∂_i of the Euclidean metric δ_{ij} on $\mathbb{R}^n$. Henceforth in abstract index expressions, indices will be raised and lowered with σ^{ij} and σ_{ij}, respectively, in accordance with the Einstein summation convention.

3. Kinematics of diffusion

We begin by introducing some additional kinematic quantities that will be required in our discussion. We define the forward and backward *generators* of the process $X^{\mathbf{i}}_t$, respectively, by

$$\mathcal{D} f(t,x) = \lim_{\delta t \to 0} \mathbf{E}_{t,x} \left[\frac{f(t+\delta t, X(t+\delta t)) - f(t, X(t))}{\delta t} \right], \tag{3.1a}$$

$$\tilde{\mathcal{D}} f(t,x) = \lim_{\delta t \to 0} \mathbf{E}_{t,x} \left[\frac{f(t, X(t)) - f(t-\delta t, X(t-\delta t))}{\delta t} \right] \tag{3.1b}$$

for an arbitrary function $f(t,x)$, where, as previously, $\mathbf{E}_{t,x}$ is the conditional expectation given that $X_t = x$. Since X_t is taken to satisfy (2.1) we deduce from Ito's formula [8] applied to $f(t, X_t)$ that these equations may be expressed in operator form as $\mathcal{D} = \partial/\partial t + b^i \nabla_i + (1/2)\sigma^{ij} \nabla_i \nabla_j$, $\tilde{\mathcal{D}} = \partial/\partial t + \tilde{b}^i \nabla_i - (1/2)\sigma^{ij} \nabla_i \nabla_j$ [2]. We further define the current and osmotic velocities v^i, u^i, respectively, according to

$$v^i = \tfrac{1}{2}(b^i + \tilde{b}^i), \qquad u^i = \tfrac{1}{2}(b^i - \tilde{b}^i). \tag{3.2}$$

In our discussion of the dynamics we shall require the *osmotic equation* [2] which relates the osmotic velocity u^i to the probability density via $u^i = (1/2)\sigma^{ij} \nabla_j \log \rho$. Under 'time' reversal $\mathcal{T}\colon t \mapsto -t$ these quantities transform as $\mathcal{T}[b^i] = -\tilde{b}^i$, $\mathcal{T}[\tilde{b}^i] = -b^i$ and hence $\mathcal{T}[u^i] = u^i$, $\mathcal{T}[v^i] = -v^i$. (The reversal of v^i under $\mathcal{T}$ is in accordance with intuition since

this velocity is identical to the Eulerian velocity field of fluid mechanics [12]; the invariance of u^i under $\mathcal{T}$ is consistent with the osmotic equation above.) Similarly for the generator we have $\mathcal{T}[\mathcal{D}] = -\tilde{\mathcal{D}}$.

The stochastic principle of least action we shall adopt is that for a Lagrangian $\mathcal{L}(t, X^i, b^i)$ and $\mathcal{L}(t, X^i(t)) := \mathcal{L}(t, X^i(t), b^i(t, X^i(t)))$ the quantity

$$\mathcal{I}(t) := \mathbf{E}_t \int_t^T \mathcal{L}(s, X^i(s))\, \mathrm{d}s \tag{3.3}$$

is stationarised with respect to variations in the forward drift b^i, for fixed t, T and a fixed initial configuration (probability density), i.e. $\delta\mathcal{I} = \mathrm{o}(\delta b^i)$ and $\delta\mathbf{E}_t = 0$. The *action* $\mathcal{A}$ is defined in a similar way except that the conditioning is also with respect to the location on $\mathcal{M}$ at a given time, i.e.

$$\mathcal{A}(t, x^i) := -\mathbf{E}_{t,x^i} \int_t^T \mathcal{L}(s, X^i(s))\, \mathrm{d}s, \tag{3.4}$$

where the sign is chosen for mathematical convenience. In both cases the integrals are taken over individual sample paths of the stochastic process. From the "tower law" for the conditional expectation [8] we have the identity $\mathbf{E}_t \circ \mathbf{E}_{t,x^i} \equiv \mathbf{E}_t$ and therefore $\mathcal{I}(t) = -\mathbf{E}_t \mathcal{A}(t, x^i)$. The functional variation of this relation is $\delta\mathcal{I} = -\mathbf{E}_t(\delta\mathcal{A})$ since our variational principle holds the initial configuration fixed, so that $\delta\mathbf{E}_t \equiv 0$. From the identity $(\mathrm{d}/\mathrm{d}t)(\mathbf{E}_t f) \equiv \mathbf{E}_t(\mathcal{D}f)$ we find that

$$\delta\mathcal{I}(t) = \mathbf{E}_t \int_t^T \mathcal{D}(\delta\mathcal{A}). \tag{3.5}$$

The definition of $\mathcal{A}$ and the primary definition (3.1a) of the generator $\mathcal{D}$ leads immediately to $\mathcal{D}\mathcal{A} = \mathcal{L}$, by applying the identity $\mathbf{E}_{t,x^i(t)} \circ \mathbf{E}_{t+\delta t, x^i(t+\delta t)} \equiv \mathbf{E}_{t,x^i(t)}$. Hence the integrand of (3.5) may be expressed as

$$\mathcal{D}(\delta\mathcal{A}) = \delta\mathcal{L} - (\delta\mathcal{D})\mathcal{A} = \delta\mathcal{L} - \delta b^i \nabla_i \mathcal{A}. \tag{3.6}$$

Expanding $\mathcal{D}$ this reads

$$\frac{\partial\mathcal{A}}{\partial t} + b^i\nabla_i\mathcal{A} + \frac{1}{2}\sigma^{ij}\nabla_i\nabla_j\mathcal{A} = \mathcal{L}, \tag{3.7}$$

which at this stage of development should be regarded as a purely kinematical statement.[4] Our purpose in the discussion of the dynamics below is to establish the relationship between b^i and the action $\mathcal{A}$. This is achieved by specifying $\mathcal{L}$ and a stochastic principle of least action.

4. Dynamics of conservative diffusion

The dynamics that we consider, which govern the time evolution of the forward drift of the diffusion, are taken to be time reversal symmetric. We shall argue that in the presence

[4] In Hamiltonian form, with $H = b^i p_i - \mathcal{L}$, we have $\partial\mathcal{A}/\partial t + (1/2)\sigma^{ij}\nabla_i\nabla_j\mathcal{A} = -H(x^i, \nabla_j\mathcal{A})$.

of a local potential there are essentially two possibilities for the dynamical structure, and in both cases we deduce the dynamics from the stochastic principle of least action $\delta\mathcal{I} = \mathrm{o}(\delta b^i)$ for a Lagrangian equal to a certain kinetic minus a local potential term $\mathcal{L} = T - V(x^i)$.

Given the requirement of time reversal symmetry the dynamical structures available, up to linear combinations, are

$$\frac{1}{2}(\mathcal{D}\tilde{b}_i + \tilde{\mathcal{D}}b_i) = -\frac{\nabla_i V}{m} \quad \text{(Case 1)}, \tag{4.1a}$$

$$\mathcal{D}b_i = -\frac{\nabla_i V}{m} = \tilde{D}\tilde{b}_i \quad \text{(Case 2)}. \tag{4.1b}$$

The dynamical symmetry under $\mathcal{T}$ is evident from the action of $\mathcal{T}$ on (4.1a) and (4.1b), using the identities for $\mathcal{T}$ derived in Section 3. Observe that the left-hand side of (4.1a) is invariant under $\mathcal{T}$, and that the second equality in (4.1b) is implied by the action of $\mathcal{T}$ on the first. Nevertheless, the overall process is time asymmetric for $\sigma^{ij} \neq 0$, due to the presence of the diffusive term in the Fokker–Planck equations (2.6a) and (2.6b).

Case 1. The situation we consider here constitutes a concise review of Nelson's stochastic mechanics [1,2]. We shall adopt the kinetic Lagrangian $T = (1/2)mb^i b_i + (1/2)m\nabla_i b^i$ (in a physical context the parameter m has the connotation of mass). This choice of kinetic Lagrangian is motivated as follows. Consider the squared Ito differential $\mathbf{E}_t[\mathrm{d}X^i\,\mathrm{d}X_i]$. Adapting the argument of Nelson[5] into the abstract index notation we write $\mathrm{d}X^i = b^i\,\mathrm{d}t + \nabla_k b^i \mathcal{W}^k - (1/2)\delta^{jl}\partial_k\Gamma^i_{jl} + \mathrm{d}w^i + \mathrm{O}(\mathrm{d}t^2)$, where $\mathcal{W}^k := \int_t^{t+\mathrm{d}t}[w^k(s) - w^k(t)]\,\mathrm{d}s$. Thus $\mathrm{d}X^i\,\mathrm{d}X_i = b^i b_i\,\mathrm{d}t^2 + 2b^i\,\mathrm{d}w_i\,\mathrm{d}t + 2\nabla_k b^i \mathcal{W}^k\,\mathrm{d}w_i - \delta^{jl}\partial_k\Gamma^i_{jl}\mathcal{W}^k\,\mathrm{d}w_i + \mathrm{d}w^i\,\mathrm{d}w_i + \mathrm{o}(\mathrm{d}t^2)$. Taking $\mathbf{E}_t$ of this relation we have $\mathbf{E}_t[2b^i\,\mathrm{d}w_i\,\mathrm{d}t] \equiv 0$. From the independent increments property for w^i [8] we deduce, for $t \leq s \leq r$, the integral identity

$$\begin{aligned}&\mathbf{E}_t[w^k(s) - w^k(t)][w_i(r) - w_i(t)]\\ &\quad = \delta^k_i(s-t) + \tfrac{1}{4}\delta^{km}\delta^{jl}\partial_j\partial_l\sigma_{im}(s-t)^2 + \mathrm{o}(s-t)^2.\end{aligned} \tag{4.2}$$

Integrating this equation with respect to s and setting $s = t + \mathrm{d}t$ we deduce the relations $\mathbf{E}_t[2\nabla_k b^i \mathcal{W}^k\,\mathrm{d}w_i] = \nabla_i b^i\,\mathrm{d}t^2 + \mathrm{o}(\mathrm{d}t^2)$, $\mathbf{E}_t[-\delta^{jl}\partial_k\Gamma^i_{jl}\mathcal{W}^k\,\mathrm{d}w_i] = -(1/2)\delta^{jl}\partial_i\Gamma^i_{jl}\,\mathrm{d}t^2 + \mathrm{o}(\mathrm{d}t^2)$, while contraction over i, k yields $\mathbf{E}_t[\mathrm{d}w^i\,\mathrm{d}w_i] = n\,\mathrm{d}t + (1/4)\delta^{im}\delta^{jl}\partial_j\partial_l\sigma_{im}\,\mathrm{d}t^2 + \mathrm{o}(\mathrm{d}t^2)$. Thus

$$\mathbf{E}_t[\mathrm{d}X^i\,\mathrm{d}X_i] = (b^i b_i + \nabla_i b^i - \tfrac{1}{3}\delta^{jk}\partial_{[i}\Gamma^i_{j]k})\,\mathrm{d}t^2 + n\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t^2). \tag{4.3}$$

Using the identity for the scalar curvature $R = -2\delta^{jk}\partial_{[i}\Gamma^i_{j]k}$ [6,9] we find

$$\mathbf{E}_t\left[\frac{1}{2}\frac{\mathrm{d}X^i}{\mathrm{d}t}\frac{\mathrm{d}X_i}{\mathrm{d}t}\right] = \frac{1}{2}b^i b_i + \frac{1}{2}\nabla_i b^i + \frac{1}{12}R + \frac{n}{2\mathrm{d}t} + \mathrm{o}(1). \tag{4.4}$$

The term $n/\mathrm{d}t$ is the same for all possible paths and so removing this term and dividing through by $\mathrm{d}t^2$ leads us to the kinetic Lagrangian stated above.[6] From this choice of $\mathcal{L}$ we

[5] See p. 45 et seq in [2].

[6] This ingenious argument is abbreviated from Nelson [2] and is originally due to Guerra (unpublished). We neglect the scalar curvature term since in most situations of interest, e.g. the study of diffusion on spin manifolds, R is constant and so does not affect the variational principle.

calculate from (3.6) that $\mathcal{D}(\delta\mathcal{A}) = \delta b^i(b_i - \nabla_i\mathcal{A}) + \nabla_i\delta b^i$. In expectation the final term on the right-hand side can be expressed as $\mathbf{E}_t[\nabla_i\delta b^i] = -\mathbf{E}_t[u_i\delta b^i]$, which follows from the osmotic equation and an integration by parts in the integral $\mathbf{E}_t[f] = \int \rho f$. Combining with (3.2) we find

$$\delta\mathcal{I} = \mathbf{E}_t \int_t^T (v^i - \nabla^i\mathcal{A})\delta b_i, \tag{4.5}$$

which implies the stochastic Hamilton–Jacobi condition $v^i = \nabla^i\mathcal{A}$. Inserting this into (3.7) then yields the *stochastic Hamilton–Jacobi equation*

$$\frac{\partial\mathcal{A}}{\partial t} + \frac{1}{2}\nabla^i\mathcal{A}\nabla_i\mathcal{A} - \frac{1}{2}u^iu_i - \nabla_iu^i + V = 0, \tag{4.6}$$

whereupon ∇_k yields the (time-symmetric) acceleration dynamics (4.1a). Observe that (4.6) is well posed both to the future and past.

Case 2. We neglect the $(1/2)\nabla_ib^i$ term present in the kinetic Lagrangian of the previous case, and thus adopt the choice $T = (1/2)mb^ib_i$. Then (3.6) implies that $\mathcal{D}(\delta\mathcal{A}) = \delta b^i(b_i - \nabla_i\mathcal{A})$ and so the variational principle leads us instead to $b^i = \sigma^{ij}\nabla_j\mathcal{A}$. Inserting this condition into (3.7) we obtain the modified stochastic Hamilton–Jacobi equation

$$\frac{\partial\mathcal{A}}{\partial t} + \frac{1}{2}\sigma^{ij}\nabla_i\nabla_j\mathcal{A} + \frac{1}{2}\nabla_i\mathcal{A}\nabla^i\mathcal{A} + V = 0. \tag{4.7}$$

This has the appearance of a *time-reversed* heat equation for the action. Therefore the solution for $\mathcal{A}$ is determined given final data, and accordingly the situation is adapted to stochastic control problems (cf. [14]). Taking ∇_k of (4.7) yields the first of the (time-symmetric) vector dynamical equations (4.1b), from which the second follows from the Fokker–Planck equation (2.6a).

5. Wave functions and complex structures

We begin with an exposition of the concept of a complex structure acting on a real vector space [4–6]. In terms of the discussion of wave dynamics to follow in Section 5.2 this vector space should be regarded as the Hilbert space of states $\mathcal{H}$ in the wave function description. Since the overlap between the study of complex Kähler geometry [6] and stochastic analysis is yet to be widely appreciated, we outline the concepts necessary for our development in some detail.

5.1. Complex structures on real spaces

Let $\mathbb{V}$ be a real vector space of dimension p. Consider a real endomorphism J of $\mathbb{V}$, whose square is minus the identity on $\mathbb{V}$, thus $J^2 = -\mathrm{id}|_{\mathbb{V}}$.[7] The determinant of this equation yields $0 \leq (\det J)^2 = (-1)^p$ and thus $p = 2n$ for positive integer n or else $p = \infty$. The operator

[7] In relativistic quantum theory, such a complex structure J provides the splitting of the *single particle Hilbert space* $\mathcal{H}$ into positive and negative frequency eigenspaces (cf. [5]).

J is said to be a *complex structure* on $\mathbb{V}$. Observe that for a given $\mathbb{V}$ the choice of J is non-unique. There exist two quite distinct notions of *complexification* of a real vector space $\mathbb{V}$. First, there is the extra ingredient of a complex structure J as above. Second, there is complexification of $\mathbb{V}$ in the sense of the mapping $\mathbb{V} \mapsto \mathbb{V} \otimes \mathbb{C} = \mathcal{H}$. If we extend the action of J to $\mathcal{H}$ by complex linearity, then J admits eigenspaces with eigenvalues $\pm\mathrm{i}$ (J is diagonalisable over $\mathbb{V} \otimes \mathbb{C}$), denoted $\mathcal{H}_\pm$, respectively. Then we have the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. In this way any vector $v \in \mathcal{H}$ can be decomposed uniquely as $v = v_+ + v_-$ in which $v_\pm = (1/2)(v \mp \mathrm{i}Jv)$ are the positive and negative parts of v. The uniqueness of this splitting follows from decomposing v above into $v_\pm$ and $w_\pm$ so that $(v-w)_+ = -(v-w)_-$, and applying J to this equation. Accordingly the action of J is given explicitly in terms of positive and negative parts by $Jv = \mathrm{i}(v_+ - v_-)$. The simplest example that illustrates this idea is $\mathbb{V} = \mathbb{R}^2$ with standard basis denoted $\{e_i\}$. Consider a complex structure represented in this basis as $J^{\mathbf{a}}{}_{\mathbf{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We can express a vector v as the sum of its positive and negative parts in this basis according to $v^{\mathbf{a}} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} z \\ -\mathrm{i}z \end{pmatrix} + \begin{pmatrix} \tilde{z} \\ \mathrm{i}\tilde{z} \end{pmatrix}$, where $z = a + \mathrm{i}b$, $\tilde{z} = a - \mathrm{i}b$. Alternatively, with respect to the diagonal basis $\{e'_i\}$ with $e'_1 = (1/2)\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}$, $e'_2 = (1/2)\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$, we have $J^{\mathbf{a}'}{}_{\mathbf{b}'} = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and effect the decomposition $v^a = v^\alpha \oplus \tilde{v}^{\alpha'}$, $v^\alpha = \begin{pmatrix} z \\ 0 \end{pmatrix}$, $\tilde{v}^{\alpha'} = \begin{pmatrix} 0 \\ \tilde{z} \end{pmatrix}$ so that unprimed (primed) abstract Greek indices denote the $\pm\mathrm{i}$ eigenspaces of J, respectively. Complex conjugation of a vector represented in the standard basis simply effects ordinary complex conjugation of its components. In the diagonal basis we have instead $v = \begin{pmatrix} z \\ \tilde{z} \end{pmatrix}$, $\bar{v} = \begin{pmatrix} \tilde{z}^* \\ z^* \end{pmatrix}$, where $*$ denotes complex conjugation in $\mathbb{C}$ and z, $\tilde{z}$ are independent complex numbers. A vector v is said to be *real* if and only if $v = \bar{v}$ and thus $v \in \mathbb{V}$, i.e. its components with respect to the standard basis are real. In the diagonal basis this is the condition that $\tilde{z} = z^*$, i.e. the positive and negative parts of v are complex conjugates of each other.

Remark 5.1. (a) The choice of J does not uniquely determine the map $v \mapsto z$, $v \in \mathbb{V}$ since there is rotational freedom in the basis for $\mathbb{V}$. (b) The complex conjugation $v \mapsto \bar{v}$ above is distinct from the *pseudo-conjugation* operation $\mathcal{C}$ given as follows. Write $\mathbb{V} = \mathbb{U} \oplus J[\mathbb{U}]$, where $\dim_{\mathbb{R}} \mathbb{U} = n$ with $J = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}$ and decompose any $v \in \mathbb{V} \otimes \mathbb{C}$ as $v = u \oplus Ju'$ with $u, u' \in \mathbb{U} \otimes \mathbb{C}$. Define the pseudo-conjugation on $\mathbb{V} \otimes \mathbb{C}$ according to $\mathcal{C}v = u - Ju'$. Accordingly, $u = (1/2)(v + \mathcal{C}v)$ and $u' = (1/2)J^{-1}(v - \mathcal{C}v)$. (c) Complex conjugation $v \mapsto \bar{v}$, $v \in \mathbb{V}$ is in general distinct from the *Hermitian* conjugation mapping $v^\alpha \mapsto \tilde{v}^{\alpha'}$ determined by J, which interchanges the positive and negative parts of any $v \in \mathcal{H}$.

5.2. Wave dynamics

Returning to the discussion of Section 4 we provide a description of the dynamical structures (4.1a) and (4.1b) in terms of wave functions. For this purpose we introduce velocity potentials such that $u^i = \nabla^i R$, $v^i = \nabla^i S$. (This is possible since the Hamilton–Jacobi

conditions in both cases ensure that u^i and v^i are curl free.) In accordance with the osmotic equation we adjust the freedom of an additive constant in R such that $\rho = \exp(2R)$. We shall set $\hbar = 2m\Sigma$ and $\sigma^{ij} = \Sigma g^{ij}$ for constant Σ and take ∇ to be Levi-Civita with respect to g.[8]

Case 1. The situation for Nelson's stochastic mechanics can be summarised as follows. Writing $\phi = \exp(R + \mathrm{i}S)$ we have $\rho = \phi\bar{\phi}$ and deduce from (4.6) that ϕ satisfies the Schrödinger equation $\mathrm{i}\hbar\partial\phi/\partial t = (-\hbar^2/2m)\nabla^2\phi + V\phi$. In the abstract index Hilbert space notation of Section 5.1 with $\phi^\alpha \leftrightarrow \exp(R + \mathrm{i}S)$, $\bar{\phi}^{\alpha'} \leftrightarrow \exp(R - \mathrm{i}S)$ and $\phi^a = \phi^\alpha \oplus \bar{\phi}^{\alpha'}$ this can be expressed as

$$\hbar J^a_b \frac{\partial \phi^b}{\partial t} = H^a_b \phi^b, \tag{5.1}$$

where H^a_b represents the Hamiltonian on $\mathcal{H}$.[9] The real part of this equation is equivalent to the Fokker–Planck equation (2.6a), and ∇_i of the imaginary part implies the acceleration dynamics (4.1a).

Case 2. We proceed in a similar way to the above. In this case, however, we introduce a pair of real valued wave functions $\psi = \exp(R + S)$, $\tilde{\psi} = \exp(R - S)$, so that $\rho = \psi\tilde{\psi}$. Then from (4.7) we deduce the pair of dynamical equations

$$-\hbar\frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\psi + V\psi, \qquad \hbar\frac{\partial \tilde{\psi}}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\tilde{\psi} + V\tilde{\psi}. \tag{5.2}$$

If we introduce $\Psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}$, i.e. $\Psi^a = \psi^\alpha \oplus \tilde{\psi}^{\alpha'} \in \mathcal{H}$ and a complex structure J acting on Ψ via $J\begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = \mathrm{i}\begin{pmatrix} -\tilde{\psi} \end{pmatrix}$ then we can re-express (5.2) as the *single* equation

$$-\mathrm{i}J\frac{\partial \Psi}{\partial t} = \left(\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi. \tag{5.3}$$

Alternatively taking appropriate linear combinations of (5.2) we find, in terms of the standard basis for $\mathbb{V}$ of Section 5.1, that

$$\frac{\partial}{\partial t}[\mathrm{i}(\psi - \tilde{\psi})] = \hat{H}[\psi + \tilde{\psi}], \qquad \frac{\partial}{\partial t}[\psi + \tilde{\psi}] = \hat{H}[-\mathrm{i}(\psi - \tilde{\psi})]. \tag{5.4}$$

Thus (5.3) holds with $J[\Psi]$ represented in this basis as $J\Psi^{\mathbf{a}} = \begin{pmatrix} \mathrm{i}(\psi - \tilde{\psi}) \\ \psi + \tilde{\psi} \end{pmatrix}$. Observe incidentally that we could also write (5.2) as $(\partial/\partial t)\mathcal{C}[\Psi] = \hat{H}\Psi$ using the conjugation operator $\mathcal{C}$ of Remark 5.1, and regarding ψ as an element of the space $\mathbb{U}$ instead of the positive eigenspace $\mathcal{H}^\alpha$. Accordingly the action of the complex structure on Ψ is changed to $J[\Psi]^{\mathbf{a}} = (-\tilde{\psi}, \psi)$, which follows from the off-diagonal representation of J.

[8] If $x \in \mathcal{M}$ has dimensions L and t has dimensions time T, then g is dimensionless and Σ has the dimensions of diffusion coefficient L^2T^{-1}.

[9] The vectors ϕ^α, $\bar{\phi}^{\alpha'}$ correspond, respectively, to the familiar Dirac 'kets' and 'bras' of the ordinary quantum theory [3].

Kinematically, from the discussion following (3.2) we have the symmetries $\mathcal{T}[R] = R$ and $\mathcal{T}[S] = -S$. Thus for Case 1 $\mathcal{T}[\phi] = \bar{\phi}$ (so that the action of $\mathcal{T}$coincides with ordinary complex conjugation), whereas for Case 2 we find $\mathcal{T}(\psi, \tilde{\psi}) = (\tilde{\psi}, \psi)$. Therefore in both cases *time reflection $\mathcal{T}$ effects Hermitian conjugation of the relevant wave functions with respect to the appropriate complex structure*. The symplectic currents that occur in both cases are kinematically equivalent and have analogous expressions in terms of the relevant wave functions. In Case 1 the symplectic current is $J_i = (\hbar/2\mathrm{i}m)(\bar{\phi}\nabla_i\phi - \phi\nabla_i\bar{\phi})$, whereas in Case 2 we have $J_i = (\hbar/2m)(\tilde{\psi}\nabla_i\psi - \psi\nabla_i\tilde{\psi})$. A straightforward calculation using the velocity potentials R, S shows that in either case $J^i = \rho v^i$, i.e. the symplectic current constructed from the wave functions coincides with the (densitised) probability current of the underlying stochastic theory.

5.2.1. Energy conservation

In Case 1 we have the ordinary quantum mechanical Hamiltonian operator $\hat{H}_{(1)} = -(\hbar^2/2m)\nabla^i\nabla_i + V$ whose operator expectation is prescribed by $\langle\hat{H}_{(1)}\rangle = \bar{\phi}_\alpha H^\alpha_{(1)\beta}\phi^\beta = \int\bar{\phi}\hat{H}_{(1)}\phi$. Expressing ϕ in terms of the velocity potentials R, S we see that stochastically this energy is equivalent to $(1/2)\mathbf{E}_t[v^iv_i + u^iu_i] + \mathbf{E}_t[V]$, which is conserved by virtue of the Schrödinger equation (5.1). On the other hand, for Case 2 we have the Hamiltonian operator $\hat{H}_{(2)} = (\hbar^2/2m)\nabla^i\nabla_i + V$ and define its expectation in the wave function formalism analogously as $\langle\hat{H}_{(2)}\rangle = \tilde{\psi}_\alpha H^\alpha_{(2)\beta}\psi^\beta = \int\tilde{\psi}\hat{H}_{(2)}\psi$. Expressing $\psi, \tilde{\psi}$ in terms of the velocity potentials R, S we deduce that

$$\langle\hat{H}_{(2)}\rangle = \tfrac{1}{2}\mathbf{E}_t[v^iv_i - u^iu_i] + \mathbf{E}_t[V], \tag{5.5}$$

and from the wave function dynamical equation (5.3) this energy expression is also conserved in time. (Note, however, that the contribution involving u^i, v^i is not positive definite.) Contrasting the two cases we observe the following 'duality' principle, that under expectation $\mathbf{E}_t$ the kinetic action of Case 1 is equal to the conserved kinetic energy of Case 2, i.e.

$$\mathbf{E}_t[T_{(1)}] = \langle\hat{H}_{(2)} - V\rangle. \tag{5.6}$$

(If instead we required the two cases to be dynamically equivalent, for *distinct* potentials $V_{(i)}$, then comparison of (4.6) and (4.7) shows that the required condition is $V_{(2)} = V_{(1)} - \hbar\nabla_iu^i - mu^iu_i$. This renders the conserved energies in both cases equal.) The reader should compare this result with p. 75 of [2] for a discussion of the Yasue versus Guerra–Morato formulations of Nelson's stochastic mechanics. In the Guerra–Morato formulation the forward drift velocity is varied with the initial configuration held fixed, and the process is required to be Markovian; for the non-Markovian case Yasue considers an action principle where the path is varied with fixed initial and final endpoints. Both of these stochastic principles of least action lead to the same dynamical law for the stochastic mean acceleration, which is equivalent to Schrödinger's equation. For a free particle the Yasue action is $I_{\mathbf{Y}} = \mathbf{E}_{t,T}\int_t^T(1/2)(v^jv_j + u^ju_j)\,\mathrm{d}s$, as compared to the Guerra–Morato action which from (4.4) is $I_{\mathbf{GM}} = \mathbf{E}_t\int_t^T(1/2)(v^jv_j - u^ju_j)\,\mathrm{d}s$. Thus as Nelson [2] remarks, the osmotic energy $(1/2)u^ju_j$ appears as kinetic energy in the Yasue formulation but as

potential energy in the Guerra–Morato formulation. (Compare also [15] for a discussion of energy conservation laws within the context of Nelson's stochastic mechanics.)

5.2.2. Stationary states

The space of stationary states in Case 1 should be familiar from the standard quantum theory [3,16]. By comparison in Case 2 the situation is essentially the same in this respect except for the transformation of the potential $V \mapsto -V$, as can be seen by separation of variables according to $\psi_n = \exp(-E_n t)\chi_n(x)$, $\tilde{\psi}_n = \exp(E_n t)\chi_n(x)$. Thus for a quadratic potential $V = -x^i x_i$, $\mathcal{M} \cong \mathbb{R}^3$ the stationary states of (5.2) can be inferred immediately from the corresponding situation for the quantum harmonic oscillator with its creation and annihilation operators [3]. Accordingly the space of coherent states can be constructed in an analogous way [5].

5.2.3. Superposition principles

Both dynamical structures exhibit interference, but of a different character. In Case 1 the interference should be familiar from ordinary quantum theory and is sinusoidal in essence. By comparison in Case 2 the interference also arises at the level of the wave functions ψ, $\tilde{\psi}$, since (5.2) are linear. However, in this case the interference is *hyperbolic* in nature. Consider solutions $\phi_j = \exp(R_j + \mathrm{i}S_j)$ of (5.1), and $\psi_j = \exp(R_j + S_j)$, $\tilde{\psi}_j = \exp(R_j - S_j)$ of (5.3), with probability densities ρ_j given by $\phi_j\bar{\phi}_j$ and $\psi_j\tilde{\psi}_j$, respectively. In Case 1 a linear superposition $\phi_3 = \phi_1 + \phi_2$ yields a probability density function for the superposed state

$$\rho_3 \propto \rho_1 + \rho_2 + 2\exp(R_1 + R_2)\cos(S_1 - S_2). \tag{5.7}$$

A similar calculation in Case 2 yields

$$\rho_3 \propto \rho_1 + \rho_2 + 2\exp(R_1 + R_2)\cosh(S_1 - S_2). \tag{5.8}$$

Thus both cases exhibit interference via the appearance of the third term in (5.7) and (5.8). The interference is circular in Case 1 and hyperbolic in Case 2, with respect to the velocity potential S for the fluid current v^i. It is the circular interference that leads to the fringes encountered in ordinary quantum mechanics and which also appear, albeit in hyperbolic form, for the second case of the dynamical structure.

Acknowledgements

The author is grateful to Brian Bramson for introducing him to the subject of stochastic control theory and for pointing out the existence of a pair of complementary real valued wave functions. We also thank Lane Hughston, John Lafferty and Edward Nelson for stimulating discussions.

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